UNIQUENESS OF p-AREA MINIMIZERS AND INTEGRABILITY OF A HORIZONTAL NORMAL IN THE HEISENBERG GROUP

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ABSTRACT. We study the uniqueness of p-minimal surfaces in the Heisenberg group. The p-area (or horizontal area) of a graph defined by u reads $\int |\nabla u+\vec{F}|$ where $\vec{F} = (-x^{1'}, x^1, -x^{2'}, x^2, ..., -x^{n'}, x^n)$. If u and v are two minimizers for the p-area satisfying the same Dirichlet boundary condition, then we can only get $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ (on the nonsingular set) where $N_{\vec{F}}(w) := \frac{\nabla w+\vec{F}}{|\nabla w+\vec{F}|}$. To conclude $u = v$ (or $\nabla u = \nabla v$), it is not straightforward as in the Riemannian case, but requires some special argument in general. In this paper, for a generalized area functional including p-area, we prove that $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ implies $\nabla u = \nabla v$ in dimension ≥ 3 under some rank condition on derivatives of $\vec{F}$ or the nonintegrability condition of contact form associated to u or v. Note that in dimension 2 ($n = 1$), the above statement is no longer true. Inspired by an equation for the horizontal normal $N_{\vec{F}}(u)$, we study the integrability for a unit vector to be the horizontal normal of a graph. We find a Codazzi-like equation together with this equation to form an integrability condition.

1. **Introduction and statement of the results**

Recall that the p-area is a special case of a more general area functional:

$$\mathcal{F}_H(u) \equiv \int_\Omega \{|\nabla u + \vec{F}| + Hu\}\ dx^1 \wedge dx^2 \wedge ... \wedge dx^m. \tag{1.1}$$

where $\Omega \subset R^m$ is a bounded domain, $u \in W^{1,1}(\Omega)$, $\vec{F}$ is an L^1 vector field on Ω, and $H \in L^\infty(\Omega)$, say. We denote $\mathcal{F}_H$ by $\mathcal{F}_0$ for the case of $H = 0$:

$$\mathcal{F}_0(u) \equiv \int_\Omega |\nabla u + \vec{F}|. \tag{1.2}$$

$\mathcal{F}_0(\cdot)$ is called the p-area (of the graph defined by u over Ω) if $\vec{F} = -\vec{X}^*$ where $\vec{X}^* = (x^{1'}, -x^1, x^{2'}, -x^2, ..., x^{n'}, -x^n)$, $m = 2n$ (see [7]). In the case of a graph Σ over the R^{2n}-hyperplane in the Heisenberg group, the above definition of p-area coincides with those given in [4], [12], and [21]. In particular these notions, especially in the framework of geometric measure theory, have been used to study existence or regularity properties of minimizers for the relative perimeter or extremizers of isoperimetric inequalities (see, e.g., [12], [14], [16], [17], [19], [20], [22]).

The p-area can also be identified with the $2n+1$-dimensional spherical Hausdorff measure of Σ (see, e.g., [2], [13]). Some authors take the viewpoint of so called intrinsic graphs (see, e.g., [13], [1], [3]). Starting from the work [7] (see also [5]), we studied the subject from the viewpoint of partial differential equations and that

1991 *Mathematics Subject Classification.* Primary: 35L80; Secondary: 35J70, 32V20, 53A10, 49Q10.

Key words and phrases. Minimizer, p-area, Heisenberg group, horizontal normal.

of differential geometry (see [9], [10], [8], and [6]; we use the term p-minimal since this is the notion of minimal surfaces in pseudohermitian geometry; "p" stands for "pseudohermitian").

First look at the integrand $D_u := |\nabla u+\vec{F}|$ in $\mathcal{F}_H$. Denote $\frac{\partial u}{\partial x_i}$ by u_i. We compute

$$\frac{\partial^2 D_u}{\partial u_i \partial u_j} = \frac{\delta_{ij}}{D_u} - \frac{(u_i + F_i)(u_j + F_j)}{D_u^3}.$$

Observe that $\frac{\partial^2 D_u}{\partial u_i \partial u_j}\xi_i\xi_j \geq 0$ (summation convention), but $\frac{\partial^2 D_u}{\partial u_i \partial u_j}\xi_i\xi_j = 0$ does not imply $\xi_i = 0$ for all i. $(\frac{\partial^2 D_u}{\partial u_i \partial u_j})$ being not positive definite causes trouble in studying $\mathcal{F}_H$. Let $S_{\vec{F}}(w)$ denote the singular set of a real function w defined on Ω, which consists of points $p \in \Omega$ such that $\nabla w + \vec{F} = 0$ at p. Let $\mathcal{F}(\varepsilon) := \mathcal{F}_H(u_\varepsilon)$ where $u_\varepsilon := u + \varepsilon\varphi$, $\varphi \in W_0^{1,1}$. We have

$$\begin{aligned} \text{(1.3)} \qquad \frac{d\mathcal{F}(0\pm)}{d\varepsilon} \quad &: = \lim_{\varepsilon\to 0\pm} \frac{\mathcal{F}(\varepsilon) - \mathcal{F}(0)}{\varepsilon} \\ &= \pm \int_{S_{\vec{F}}(u)} |\nabla\varphi| + \int_{\Omega \setminus S_{\vec{F}}(u)} N_{\vec{F}}(u)\cdot\nabla\varphi + \int_\Omega H\varphi \end{aligned}$$

((3.3) with $\hat{\varepsilon} = 0$ in [9]) where we denote the horizontal normal $\frac{\nabla w+\vec{F}}{|\nabla w+\vec{F}|}$ for a real function w ($\in W^1(\Omega)$, say) by $N_{\vec{F}}(w)$ (or ν^w; the notation $N_{\vec{F}}(w)$ has been used previously. But the notation ν^w is concise). From (1.3), the first variation formula of $\mathcal{F}_H$ at u, we found that $\pm\int_{S_{\vec{F}}(u)} |\nabla\varphi|$ is not negligible if $H_m(S_{\vec{F}}(u)) \neq 0$. In [6], we extended the range of $u \in W^{1,1}$ to $u \in BV$ and computed the first and second variations. For $u, v \in BV$, $u_\varepsilon = u + \varepsilon\varphi$ with $\varphi = v - u$, the right and left derivatives $\mathcal{F}'_\pm(\varepsilon) := \lim_{\tilde{\varepsilon}\to\varepsilon\pm} \frac{\mathcal{F}(\tilde{\varepsilon})-\mathcal{F}(\varepsilon)}{\tilde{\varepsilon}-\varepsilon}$ exist and satisfy

$$\mathcal{F}'_-(\varepsilon_1) \leq \mathcal{F}'_+(\varepsilon_1) \leq \mathcal{F}'_-(\varepsilon_2) \leq \mathcal{F}'_+(\varepsilon_2)$$

for $\varepsilon_1 < \varepsilon_2$ ((3.20) in [6]). We also have

$$\lim_{\varepsilon_2\to\varepsilon_1+} \mathcal{F}'_\pm(\varepsilon_2) = \mathcal{F}'_+(\varepsilon_1), \quad \lim_{\varepsilon_2\to\varepsilon_1-} \mathcal{F}'_\pm(\varepsilon_2) = \mathcal{F}'_-(\varepsilon_1)$$

and $\mathcal{F}$ is convex in ε ((3.21) in [6]). For the second variation, although $\mathcal{F}'_+(\varepsilon)$ may not equal $\mathcal{F}'_-(\varepsilon)$, we have

$$\lim_{\varepsilon_2\to\varepsilon_1+} \frac{\mathcal{F}'_\pm(\varepsilon_2) - \mathcal{F}'_+(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1} = \lim_{\varepsilon_2\to\varepsilon_1-} \frac{\mathcal{F}'_\pm(\varepsilon_2) - \mathcal{F}'_-(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1}$$

(Theorem C in [6]). That is to say, the first variation may have jumps, but the right and left limits of the second variation exist and coincide. This is an interesting property.

In [9], we proved the uniqueness of (p-) minimizers for the p-area (or horizontal area) in the space $W^{1,2}$ among other things. Let $\vec{F}^* := (F_2, -F_1, F_4, -F_3, ..., F_{2n}, -F_{2n-1})$ for $\vec{F} = (F_1, F_2, ..., F_{2n})$.

Theorem 1.1 (Theorem B in [9]**).** *Let Ω be a bounded domain in R^{2n}. Let $u, v \in W^{1,2}(\Omega)$ be two minimizers for $\mathcal{F}_H$ such that*

$$u - v \in W_0^{1,2}(\Omega). \tag{1.4}$$

Suppose $H \in L^{\infty}(\Omega)$ *and* $\vec{F} \in W^{1,2}(\Omega)$ *satisfying*

$$\text{div}\, \tilde{F}^{*} > 0 \text{ a.e. } (\text{div}\, \tilde{F}^{*} < 0 \text{ a.e., resp.}) \tag{1.5}$$

Then $u \equiv v$ *in* Ω *(a.e.).*

The uniqueness of BV solutions to the appropriate Dirichlet problem is still unknown. However for $u, v \in W^{1,2}$ as in Theorem 1.1 (Theorem B in [9]), since $\mathcal{F}(\varepsilon)$ is nondecreasing and $\mathcal{F}(0) = \mathcal{F}_H(u) = \mathcal{F}_H(v) = \mathcal{F}(1)$ (u, v being minimizers for $\mathcal{F}_H$), we have $\mathcal{F}(\varepsilon) = \mathcal{F}(0) = \mathcal{F}(1)$ for all ε, $0 \leq \varepsilon \leq 1$. Moreover, we can show that there are at most countably many ε such that $H_m(S_{\vec{F}}(u_\varepsilon)) \neq 0$. Choose ε_1, ε_2 $\in (0,1)$ with $H_m(S_{\vec{F}}(u_{\varepsilon_1})) = H_m(S_{\vec{F}}(u_{\varepsilon_2})) = 0$. We then have $\mathcal{F}(\varepsilon_1) = \mathcal{F}(\varepsilon_2)$ and $\mathcal{F}'(\varepsilon_1) = \mathcal{F}'(\varepsilon_2) = 0$. It follows from (1.3) that

$$\begin{aligned} 0 &= \mathcal{F}'(\varepsilon_2) - \mathcal{F}'(\varepsilon_1) \\ &= \int_\Omega (N_{\vec{F}}(u_{\varepsilon_2}) - N_{\vec{F}}(u_{\varepsilon_1})) \cdot \frac{\nabla u_{\varepsilon_2} - \nabla u_{\varepsilon_1}}{\varepsilon_2 - \varepsilon_1}. \end{aligned}$$

Here we have used $\varphi = v - u = \frac{u_{\varepsilon_2} - u_{\varepsilon_1}}{\varepsilon_2 - \varepsilon_1}$. Now the equality $(N_{\vec{F}}(u_{\varepsilon_2}) - N_{\vec{F}}(u_{\varepsilon_1})) \cdot (\nabla u_{\varepsilon_2} - \nabla u_{\varepsilon_1}) = \frac{1}{2}(D_{u_{\varepsilon_2}} + D_{u_{\varepsilon_1}})|N_{\vec{F}}(u_{\varepsilon_2}) - N_{\vec{F}}(u_{\varepsilon_1})|^2$ (cf. Lemma 5.1$'$ in [7]) forces that $N_{\vec{F}}(u_{\varepsilon_1}) = N_{\vec{F}}(u_{\varepsilon_2})$ (a.e.).

Note that having made use of the boundary condition (1.4), we prove $N_{\vec{F}}(u_{\varepsilon_1}) = N_{\vec{F}}(u_{\varepsilon_2})$ for so called regular ε_1, $\varepsilon_2 \in [0,1]$ (see Section 3 or [9] for more detail), in which $u_\varepsilon := u + \varepsilon(v - u)$ (in the case of good regularity, we have $N_{\vec{F}}(u) = N_{\vec{F}}(v)$). In fact, the difficulty of the proof of Theorem 1.1 is that we may have $H_{2n}(S_{\vec{F}}(u))$ $\neq 0$ or $H_{2n}(S_{\vec{F}}(v)) \neq 0$. We avoid such difficulty by working on regular ε. Next together with the condition (1.5) we can show $u \equiv v$ (in particular, $\nabla u = \nabla v$) in Ω (a.e.).

In this paper we will first focus on the problem when $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ implies $\nabla u = \nabla v$ with no boundary condition (1.4). In general this is not possible. For instance, $u = xy$ and $v = xy + y$ in the Heisenberg group of dimension 3. See Example 2.2 for details. On the positive side, we find a rank condition on the derivatives of $\vec{F}$. Let $h_{IJ} := \partial_I F_J - \partial_J F_I$ (see (2.6)). The rank of a matrix A, denoted as $rank(A)$, is the dimension of the range $Range(A)$ (or image) of A. Note that for all the results below in this paper, we do not assume $m = 2n$.

Theorem A. *Let* u, $v \in C^2(\Omega)$ *and* $\vec{F} \in C^1(\Omega)$ *where* Ω *is a domain in* R^m. *Suppose both* $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ *and* $\nabla N_{\vec{F}}(u) = \nabla N_{\vec{F}}(v)$ *at one point* $p \in \Omega \setminus [S_{\vec{F}}(u)$ $\cup\, S_{\vec{F}}(v)]$. *Assume*

$$m \ \geq \ rank(h_{IJ}(p)) \ \geq \ 3.$$

Then $\nabla u = \nabla v$ *at* $p \in \Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$.

By adding the boundary condition we then have the uniqueness of minimizers for $\mathcal{F}_H$.

Corollary A.1. *Let* Ω *be a bounded domain in* R^m. *Take* $\vec{F} \in C^1(\bar{\Omega})$ *and* H $\in L^{\infty}(\Omega)$. *Let* u, $v \in C^2(\Omega) \cap C^0(\bar{\Omega}) \cap W^{1,1}(\Omega)$ *be two minimizers for* $\mathcal{F}_H(\cdot)$ *in* $W^{1,1}(\Omega)$ *(see (1.1) and Definition 3.1 in* [9]*) with* $u = v$ *on* $\partial\Omega$. *Suppose* $m \geq$ $rank(h_{IJ}(p)) \geq 3$ *for all* $p \in \Omega$. *Then* $u = v$ *in* Ω.

A weak version of Theorem A (Corollary A.1, resp.) reads as follows:

Theorem A′. *Let u, $v \in W^2(\Omega)$ and $\vec{F} \in W^1(\Omega)$ where Ω is a domain in R^m. Suppose for some constant $C > 0$, $|\nabla u + \vec{F}| \geq C$, $|\nabla v + \vec{F}| \geq C$, $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in Ω (a.e.). Assume*

$$m \ \geq \ rank(h_{IJ}) \ \geq \ 3.$$

in Ω (a.e.). Then $\nabla u = \nabla v$ in Ω (a.e.).

Corollary A′.1. *Let Ω be a bounded domain in R^m. Take $\vec{F} \in W^1(\bar{\Omega})$ and $H \in L^\infty(\Omega)$. Let u, $v \in W^{2,1}(\Omega)$ be two minimizers for $\mathcal{F}_H(\cdot)$ in $W^{1,1}(\Omega)$ (see (1.1) and Definition 3.1 in* [9]*) with $u - v \in W_0^{2,1}(\Omega)$. Suppose for some constant $C > 0$, $|\nabla u + \vec{F}| \geq C$, $|\nabla v + \vec{F}| \geq C$. Suppose $m \geq rank(h_{IJ}) \geq 3$ in Ω (a.e.). Then $u = v$ in Ω (a.e.).*

Next we find a nonintegrability condition for $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ to imply $\nabla u = \nabla v$. Let $\Theta_w := dw + F_I dx_I$ for a real function w defined on Ω. If the distribution defined by $\Theta_w = 0$ in Ω is integrable, then we have

$$\Theta_w \wedge d\Theta_w = 0$$

We say Θ_w is integrable (nonintegrable, respectively) at a point $p \in \Omega$ if $\Theta_w \wedge d\Theta_w = 0$ ($\Theta_w \wedge d\Theta_w \neq 0$, respectively) at p. The integrability condition can be described in terms of h_{IJ} and $\nu_K^w := (\partial_K w + F_K)/|\nabla w + \vec{F}|$ (see (2.18); note that $N_{\vec{F}}(w) = \nu^w$). For $w \in W^2(\Omega)$ and $\vec{F} \ \in \ W^1(\Omega)$, we say Θ_w is nonintegrable if $\Theta_w \wedge d\Theta_w \neq 0$ in Ω a.e..

Theorem B. *Let u, $v \in C^2(\Omega)$ and $\vec{F} \in C^1(\Omega)$ where Ω is a domain in R^m. Suppose $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$. Suppose for each point in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$, either Θ_u is nonintegrable or Θ_v is nonintegrable. Then $\nabla u = \nabla v$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$.*

Again by adding the boundary condition we then have the uniqueness of minimizers for $\mathcal{F}_H$.

Corollary B.1. *Let Ω be a bounded domain in R^m. Take $\vec{F} \in C^1(\bar{\Omega})$ and $H \in L^\infty(\Omega)$. Let u, $v \in C^2(\Omega) \cap C^0(\bar{\Omega}) \cap W^{1,1}(\Omega)$ be two minimizers for $\mathcal{F}_H(\cdot)$ in $W^{1,1}(\Omega)$ (see Definition 3.1 in* [9]*) with $u = v$ on $\partial\Omega$. Suppose for each point in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$, either Θ_u is nonintegrable or Θ_v is nonintegrable. Then $u = v$ in Ω.*

A weak version of Theorem B (Corollary B.1, resp.) reads as follows:

Theorem B′. *Let u, $v \in W^2(\Omega)$ and $\vec{F} \in W^1(\Omega)$ where Ω is a domain in R^m. Suppose for some constant $C > 0$, $|\nabla u + \vec{F}| \geq C$, $|\nabla v + \vec{F}| \geq C$, $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in Ω (a.e.). Suppose either Θ_u is nonintegrable or Θ_v is nonintegrable. Then $\nabla u = \nabla v$ in Ω (a.e.).*

Corollary B′.1. *Let Ω be a bounded domain in R^m. Take $\vec{F} \in W^1(\Omega)$ and $H \in L^\infty(\Omega)$. Let u, $v \in W^{2,1}(\Omega)$ be two minimizers for $\mathcal{F}_H(\cdot)$ in $W^{1,1}(\Omega)$ (see Definition 3.1 in* [9]*) with $u - v \in W_0^{2,1}(\Omega)$. Suppose for some constant $C > 0$, $|\nabla u + \vec{F}| \geq C$, $|\nabla v + \vec{F}| \geq C$ (a.e.). Suppose either Θ_u is nonintegrable or Θ_v is nonintegrable. Then $u = v$ in Ω (a.e.).*

Note that in the above results the dimension "m" is not necessarily even. We can also extend Theorem 1.1 under a condition more general than div $\vec{F}^* >$ (or $<$) 0 while the dimension "m" is not necessarily even. Define $\vec{G}^b$ for $\vec{G} = (G_1, ..., G_m)$ by

$$\vec{G}^b := (\sum_{k=1}^{m} a^{1k} G_k, \sum_{k=1}^{m} a^{2k} G_k, ..., \sum_{k=1}^{m} a^{mk} G_k)$$

where $a^{jk\prime}s$ are real constants such that $a^{jk} + a^{kj} = 0$ for $1 \le j, k \le m$. Note that $\vec{G}^b = \vec{G}^*$ for $m = 2n$, $a^{2j-1,2j} = -a^{2j,2j-1} = 1$, $1 \le j \le n$, $a^{jk} = 0$ otherwise.

Theorem C. *Let Ω be a bounded domain in R^m. Let u, $v \in W^{1,2}(\Omega)$ be two minimizers for $\mathcal{F}_H$ such that $u - v \in W_0^{1,2}(\Omega)$. Suppose $H \in L^\infty(\Omega)$ and $\vec{F} \in W^{1,2}(\Omega)$ satisfying*

$$\text{div}\, \vec{F}^b = \sum_{j,k=1}^{m} a^{jk} \partial_j F_k > 0 \ (< 0, \ \text{resp.}) \ \text{(a.e.)}$$

where a^{jk}'s are real constants such that $a^{jk} + a^{kj} = 0$. Then $u \equiv v$ in Ω (a.e.).

Compare (2.7) with the Euclidean situation: $\delta_I \nu_J - \delta_J \nu_I = 0$ where

$$\nu = (\nu_I) = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}$$

denotes the unit normal to the graph defined by u. It is a known fact that ν can be realized as the unit normal vectors of a family of (hyper)surfaces filling up a region if and only if $\delta_I \nu_J - \delta_J \nu_I = 0$ (see page 3 in [18]).

Recall that in our situation, the horizontal normal ν^u of a graph defined by u reads

$$\nu^u = \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}.$$

Let ν_K^u denote the components of ν^u. Recall that the horizontal tangential operator δ_K^u is defined by $\delta_K^u = \partial_K - \nu_K^u \nu_J^u \partial_J$ (cf. (2.1)). Recall that $D_u := |\nabla u + \vec{F}|$ and $h_{IJ} := \partial_I F_J - \partial_J F_I$ (cf. (2.6)). In Section 2, we deduce

$$\begin{aligned} & \delta_I^u \nu_J^u - \delta_J^u \nu_I^u \\ = \ & \frac{1}{D_u}\{h_{IJ} - \nu_J^u \nu_K^u h_{IK} - \nu_I^u \nu_K^u h_{KJ}\}. \end{aligned}$$

(i.e., (2.7)). In view of the Euclidean situation, we ask the following question:

Question D: Given $D > 0$, a unit vector $\nu = (\nu_J)$, $\vec{F} = (F_J)$ locally in R^m satisfying

$$\begin{aligned} & \delta_I \nu_J - \delta_J \nu_I \\ = \ & \frac{1}{D}\{h_{IJ} - \nu_J \nu_K h_{IK} - \nu_I \nu_K h_{KJ}\}, \end{aligned} \tag{1.6}$$

can we find u such that $\nu = \frac{\nabla u + \vec{F}}{D}$?

Here $\delta_J := \partial_J - \nu_J \nu_K \partial_K$. In fact, we are asking if (1.6) is an integrability condition for ν to be the horizontal normal of a graph defined by u. It turns out

that we need a condition other than (1.6) to conclude $\nu = \frac{\nabla u + \vec{F}}{D}$. Denote 1-forms $\nu_J dx^J$, $F_J dx^J$ by $\nu^{\#}$, $F^{\#}$, resp.. When $\nu = \frac{\nabla u+\vec{F}}{D}$, we get $D\nu^{\#} = du + F^{\#}$. It follows that $d(D\nu^{\#}) = dF^{\#}$. So we have

$$\nu^{\#} \lrcorner d(D\nu^{\#} - F^{\#}) = 0. \tag{1.7}$$

Recall that the interior product $\eta \lrcorner \omega$ of 1-form η and 2-form ω is defined to be $(\eta_{\#}) \lrcorner \omega := \omega(\eta_{\#})$ where $\eta_{\#}$ is the corresponding vector of η with respect to the Euclidean metric. In practice, we have $dx^i \lrcorner (dx^j \wedge dx^k) = < dx^i, dx^j > dx^k - < dx^i, dx^k > dx^j = \delta^{ij} dx^k - \delta^{ik} dx^j$ (δ^{ij} denotes the Kronecker delta, i.e., $\delta^{ij} = 1$ if $i = j$; $\delta^{ij} = 0$ if $i \neq j$), and hence $(\eta_i dx^i) \lrcorner (\omega_{jk} dx^j \wedge dx^k) = \eta_i \omega_{jk} \{dx^i \lrcorner (dx^j \wedge dx^k)\} = \eta_j \omega_{jk} dx^k - \eta_k \omega_{jk} dx^j = \eta_j(\omega_{jk} - \omega_{kj}) dx^k$.

We can view (1.7) as a system of first order equations in D coupled with (1.6), a system of first order equations in ν. It is not hard to rewrite (1.7) as follows:

$$\delta_K D = \nu_J(\partial_J \nu_K) D - \nu_J h_{JK} \tag{1.8}$$

for any K (see (4.9) in Section 4). From the above discussion we learn that (1.6) and (1.7) (or equivalently, (1.8)) are two necessary conditions for ν to be the horizontal normal associated to a function u, i.e., $\nu = \frac{\nabla u+\vec{F}}{D}$. Conversely, they are also sufficient as we answer Question D in the following integrability theorem. For simplicity, we work in C^∞ category for this problem.

Theorem E. *Given (C^∞ smooth) $D > 0$, a unit vector $\nu = (\nu_J)$, $\vec{F} = (F_J)$ locally in R^m, $m \geq 2$, satisfying (1.6), and (1.7) or equivalently (1.8), we can find a (C^∞ smooth) function u locally such that $\nu = \frac{\nabla u+\vec{F}}{D}$.*

If $m = 2$, (1.7) or (1.8) is equivalent to $d(D\nu^{\#} - F^{\#}) = 0$ (see Proposition 4.1). However, for higher dimensions, we can have the situation that (1.7) holds while $d(D\nu^{\#} - F^{\#}) \neq 0$ (see Example 4.2). Also we can have the situation that (1.6) holds while (1.7) does not hold (see Example 4.3).

Comparing with the fundamental theorem for surfaces in the 3-dimensional Heisenberg group in [8], we don't prescribe p-mean curvature H here, but prescribe arbitrary $\vec{F}$ instead of fixed $\vec{F} = (-y, x)$ in [8]. Equation (1.7) or (1.8) corresponds to a Codazzi-like equation (cf. (1.17) in [8]). See (4.15) and the discussion before Example 4.2 in Section 4 (see also a recent preprint of Hung-Lin Chiu [11]).

The idea of proof for Theorem E is to show that $U_I := D\nu_I - F_I$ satisfy the integrability condition $\partial_I U_J = \partial_J U_I$ (and hence $U_I = \partial_I u$ for some function u). Let $U_{IJ} := \partial_I U_J - \partial_J U_I$. A direct computation shows that

$$U_{IJ} - \nu_J \nu_K U_{IK} - \nu_I \nu_K U_{KJ} = 0 \tag{1.9}$$

due to condition (1.6). Observe that in terms of differential forms, we can write (1.9) as follows:

$$\begin{aligned} d(U_I dx^I) &= \frac{1}{2} U_{IJ} dx^I \wedge dx^J \\ &= (U\nu)^{\#} \wedge \nu^{\#} \end{aligned} \tag{1.10}$$

where U denotes the matrix (U_{IJ}) and ν is viewed as a column vector in $U\nu$. We then observe that $D\nu^{\#} - F^{\#} = U_I dx^I$, and hence $d(D\nu^{\#} - F^{\#}) = \frac{1}{2} U_{IJ} dx^I \wedge dx^J$. Substituting (1.10) into condition (1.7) (or (1.8)) gives $(U\nu)^{\#} = 0$. By (1.10) again, we get $U_{IJ} = 0$, i.e., $\partial_I U_J = \partial_J U_I$.

2. **Proofs of Theorems A and B**

Recall in [9] that $S_{\vec{F}}(u) := \{p \in \Omega \mid (\nabla u + \vec{F})(p) = 0\}$ and $N_{\vec{F}}(u) := \frac{\nabla u+\vec{F}}{|\nabla u+\vec{F}|}$. The idea of the proof for the uniqueness in [9] is to show that $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$ (a.e.) first for two minimizers u, v, say, in $W^{1,1}(\Omega)$ such that $u - v \in W_0^{1,1}(\Omega)$. Then to show that $\nabla u = \nabla v$ (and hence $u = v$), we invoke an equality (see (5.3) in [9]) and an argument of integrating by parts (see Theorem 5.3 in [9]). To make this approach work, we need to assume $m = 2n$ and div $\vec{F}^* >$ (or $<$, resp.) 0 (a.e.). In this section, we are going to give another approach to show that $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ implies $\nabla u = \nabla v$. Note that in this approach, we do not need to assume $m = 2n$.

To explain the idea, we assume u, $v \in C^2$. Write $\nabla u = (u_K)$ where $u_K := \partial_K u$, $\partial_K := \frac{\partial}{\partial x_K}$, $K = 1, 2, ..., m$ and $\vec{F} = (F_K)$. So we can write

$$N_{\vec{F}}(u) = (\frac{u_K + F_K}{D_u})$$

where $D_u := |\nabla u + \vec{F}|$. Let $\hat{u}_K := u_K + F_K$ and $\nu^u_K := \frac{\hat{u}_K}{D_u}$, components of $N(u)$. Define the horizontal tangential operator δ^u_K by

$$\delta^u_K = \partial_K - \nu^u_K \nu^u_J \partial_J \tag{2.1}$$

(summing over J; summation convention hereafter). We compute

$$\partial_I \nu^u_J = \frac{\partial_I \hat{u}_J}{D_u} - \frac{\hat{u}_J \hat{u}_K \partial_I \hat{u}_K}{D_u^3}. \tag{2.2}$$

Hence from (2.1), (2.2), and the definition of ν^u_J, we have

$$\begin{aligned} \delta^u_I \nu^u_J &= \partial_I \nu^u_J - \nu^u_I \nu^u_K \partial_K \nu^u_J \\ &= \frac{\partial_I \hat{u}_J}{D_u} - \frac{\hat{u}_J \hat{u}_K \partial_I \hat{u}_K}{D_u^3} - \frac{\hat{u}_I \hat{u}_K \partial_K \hat{u}_J}{D_u^3} \\ &\quad + \frac{\hat{u}_I \hat{u}_K \hat{u}_J \hat{u}_L \partial_K \hat{u}_L}{D_u^5}. \end{aligned} \tag{2.3}$$

We can now compute

$$\begin{aligned} &\delta^u_I \nu^u_J - \delta^u_J \nu^u_I \\ = &\frac{\partial_I \hat{u}_J - \partial_J \hat{u}_I}{D_u} - \frac{\hat{u}_J \hat{u}_K (\partial_I \hat{u}_K - \partial_K \hat{u}_I)}{D_u^3} \\ &- \frac{\hat{u}_I \hat{u}_K (\partial_K \hat{u}_J - \partial_J \hat{u}_K)}{D_u^3} \end{aligned} \tag{2.4}$$

by (2.3) and noting that the term involving D_u^5 is symmetric in I, J. From the definition of $\hat{u}_J$ we have

$$\begin{aligned} &\partial_I \hat{u}_J - \partial_J \hat{u}_I \\ &= \partial_I (u_J + F_J) - \partial_J (u_I + F_I) \\ &= \partial_I F_J - \partial_J F_I. \end{aligned} \tag{2.5}$$

Let

$$h_{IJ} := \partial_I F_J - \partial_J F_I. \tag{2.6}$$

So in view of (2.5) and (2.6), we can write (2.4) as follows:

$$\begin{aligned}(2.7)\qquad & \delta_I^u \nu_J^u - \delta_J^u \nu_I^u \\ = \ & \frac{1}{D_u}\{h_{IJ} - \nu_J^u \nu_K^u h_{IK} - \nu_I^u \nu_K^u h_{KJ}\}.\end{aligned}$$

Now suppose $N_{\vec{F}}(u) = N_{\vec{F}}(v)$. Then $\nu_K^u = \nu_K^v$ and $\delta_K^u = \delta_K^v$ from the definition. It follows from (2.7) that

$$\{h_{IJ} - \nu_J^u \nu_K^u h_{IK} - \nu_I^u \nu_K^u h_{KJ}\}\left(\frac{1}{D_u} - \frac{1}{D_v}\right) = 0 \tag{2.8}$$

(in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$). If $\nabla u \neq \nabla v$, then $D_u \neq D_v$. Therefore we have

$$h_{IJ} - \nu_J^u \nu_K^u h_{IK} - \nu_I^u \nu_K^u h_{KJ} = 0 \tag{2.9}$$

by (2.8). Observe that $h = (h_{IJ})$ is a skew-symmetric matrix by (2.6), i.e.

$$h + h^T = 0 \text{ or } h_{IJ} + h_{JI} = 0 \tag{2.10}$$

where h^T denotes the transpose of h.

Lemma 2.1. *Suppose h is a skew-symmetric real $m \times m$ matrix ($m \geq 2$) such that*

$$h - h\nu\nu^T - \nu\nu^T h = 0 \tag{2.11}$$

where ν is a ($m \times 1$) unit column real vector and ν^T is the transpose of ν, a ($1 \times m$) unit row vector. Then we have

$$rank(h) = 0 \text{ or } 2. \tag{2.12}$$

where $rank(h)$ denotes the rank of h.

Proof. Multiplying (2.11) by h and then taking the trace, we obtain

$$\begin{aligned}(2.13)\qquad Tr(h^2) &= Tr(h\nu\nu^T h) + Tr(\nu\nu^T hh) \\ &= 2Tr(h\nu\nu^T h) \text{ (since } Tr(\nu\nu^T hh) = Tr(h\nu\nu^T h)) \\ &= -2||h\nu||^2 \text{ (since } h^T = -h \text{ by (2.10))}.\end{aligned}$$

Here $||\cdot||$ denotes the Euclidean norm. Observe that the eigenvalues of h (being skew-symmetric) are purely imaginary and if $i\lambda$ ($\lambda \in R\backslash\{0\}$) is a nonzero eigenvalue (with an eigenvector w), then $-i\lambda$ is also an eigenvalue (with an eigenvector independent of w). It follows that h^2 has an eigenvalue $-\lambda^2$ of multiplicity 2. Let $i\lambda_1$, $i\lambda_2$, ..,.$i\lambda_k$ ($\lambda_j \in R\backslash\{0\}$), $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_k| > 0$, $2k \leq m$, be all nonzero eigenvalues of h (if $h \neq 0$) while $-\lambda_j^2$, $j = 1, 2, ..., k$, are all nonzero eigenvalues of h^2 (each of which has multiplicity 2). From (2.13) we can easily get

$$2\lambda_1^2 \leq 2\sum_{j=1}^{k} \lambda_j^2 = 2||h\nu||^2 \leq 2\lambda_1^2.$$

Therefore $k = 1$ and hence (2.12) follows.

□

Proof. **(of Theorem A)** This follows from the above discussion. Take $h = (h_{IJ}(p))$ where $h_{IJ} = \partial_I F_J - \partial_J F_I$ and $\nu = (\nu_J^u(p))$ in Lemma 2.1. There holds (2.11) if $\nabla u \neq \nabla v$ at p by (2.8). Now the conclusion of Lemma 2.1 contradicts the assumption on $rank(h)$.

□

We remark that $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ (in a region) does not imply $\nabla u = \nabla v$ in dimension 2 as shown in the following example.

Example 2.2. Take $u = xy$ and $v = xy + y$ which define graphs over the xy-plane in the Heisenberg group of dimension 3. So in this situation we have $\vec{F} = (-y, x)$. It is straightforward to compute $\nabla u = (y, x)$, $\nabla v = (y, x + 1)$, $\nabla u + \vec{F} = (0, 2x)$, $\nabla v + \vec{F} = (0, 2x + 1)$. We observe that $N_{\vec{F}}(u) = N_{\vec{F}}(v) = (0, 1)$ in the region $\{x > 0\}$. On the other hand, it is clear that $\nabla u \neq \nabla v$.

Proof. **(of Corollary A.1)** By Theorem 5.1 in [9], we have $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$ (taking regular values $\varepsilon_j \to 0$ and 1, resp.). Then it follows from Theorem A that

$$\nabla u = \nabla v \text{ in } \Omega \setminus [S_{\vec{F}}(u) \ \cup \ S_{\vec{F}}(v)]. \tag{2.14}$$

We claim that both $S_{\vec{F}}(u)$ and $S_{\vec{F}}(v)$ are nowhere dense in Ω. Suppose the converse holds. Then we can find a small ball B contained in either $S_{\vec{F}}(u)$ or $S_{\vec{F}}(v)$, say $B \subset S_{\vec{F}}(u)$. This means that $\nabla u + \vec{F} = 0$ in B. It follows that $F_I = -u_I$ and hence $\partial_I F_J = -\partial_I u_J = -\partial_J u_I = \partial_J F_I$. Therefore $h_{IJ} = \partial_I F_J - \partial_J F_I = 0$ in B for all I, J, contradicting the condition on the rank of (h_{IJ}). The above argument also works for $B \subset S_{\vec{F}}(v)$. So we have shown that both $S_{\vec{F}}(u)$ and $S_{\vec{F}}(v)$ are nowhere dense in Ω. It follows that $S_{\vec{F}}(u) \cup S_{\vec{F}}(v)$ is nowhere dense in Ω. Therefore by (2.14) we have $u - v = c$, a constant in Ω. Since u and v are continous up to the boundary $\partial\Omega$ and $u = v$ on $\partial\Omega$, we have $c = 0$.

□

We remark that for $m = 2$, $\vec{F} \in C^1(\Omega)$ and $w \in C^1(\Omega)$, $S_{\vec{F}}(w)$ is nowhere dense in Ω if div $\vec{F}^* > 0$ (or < 0, resp.) in Ω (cf. Lemma 3.1 in [8]; in fact, we can extend this result to $m = 2n$ or even general dimensions, see Proposition 3.4 in this paper). Also note that the size of the singular set can be measured in terms of the rank of (h_{IJ}) (see Theorem D in [9] where we need to assume $u \in C^2$, $\vec{F} \in C^1$ in view of Balogh's $C^{1,1}$ examples in [2]).

We can interpret (2.9) as an integrability condition for hypersurfaces annihilated by the one-form

$$\begin{aligned} \Theta_u \quad : \quad &= du + F_I dx^I \\ &= \ (u_I + F_I)dx^I \\ &= \ D_u \nu_I^u dx^I. \end{aligned} \tag{2.15}$$

Lemma 2.3. *Let $u \in C^2(\Omega)$ and $\vec{F} \in C^1(\Omega)$ where Ω is a domain of R^m. Then in $\Omega \setminus S_{\vec{F}}(u)$, Θ_u is integrable (meaning the distribution defined by $\Theta_u = 0$ is integrable) if and only if (2.9) holds.*

Proof. Observe that Θ_u is integrable if and only if $\Theta_u \wedge d\Theta_u = 0$ by Frobenius' integrability theorem. We then compute

$$\begin{aligned} d\Theta_u &= d(du + F_I dx^I) \\ &= \partial_J F_I dx^J \wedge dx^I \\ &= \frac{1}{2} h_{IJ} dx^I \wedge dx^J \end{aligned} \tag{2.16}$$

(recall that $h_{IJ} := \partial_I F_J - \partial_J F_I$) and

$$\begin{aligned} &\Theta_u \wedge d\Theta_u \\ = \ &\frac{1}{2} D_u \nu_K^u h_{IJ} dx^K \wedge dx^I \wedge dx^J \end{aligned} \tag{2.17}$$

by (2.15) and (2.16). So $\Theta_u \wedge d\Theta_u = 0$ if and only if

$$\nu_K^u h_{IJ} + \nu_I^u h_{JK} + \nu_J^u h_{KI} = 0 \tag{2.18}$$

by (2.17) and noting that $h_{IJ} = -h_{JI}$. Multiplying (2.18) by ν_K^u and summing over K, we obtain (2.9). Conversely if (2.9) holds, we can also easily deduce (2.18). □

Proof. **(of Theorem B)**

From $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$ (comparing with the proof of Theorem A), we know that $\nabla u + \vec{F}$ is parallel to $\nabla v + \vec{F}$. It follows that in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$, there holds

$$du + F_I dx^I = \lambda(dv + F_I dx^I) \tag{2.19}$$

for some nonzero function λ. Subtracting $dv + F_I dx^I$ from (2.19) gives

$$d(u - v) = (\lambda - 1)(dv + F_I dx^I). \tag{2.20}$$

Taking exterior differentiation of (2.20), we obtain

$$0 = d\lambda \wedge \Theta_v + (\lambda - 1) d\Theta_v \tag{2.21}$$

where $\Theta_v := dv + F_I dx_I$. Wedging (2.21) with Θ_v we get

$$(\lambda - 1)\Theta_v \wedge d\Theta_v = 0. \tag{2.22}$$

Observe that $\lambda = 1$ if and only if $\nabla u = \nabla v$. So if $\nabla u \neq \nabla v$, we have $\Theta_v \wedge d\Theta_v = 0$ (and $\Theta_u \wedge d\Theta_u = 0$, resp.) by (2.22) (an identity replacing v by u, resp.). This contradicts the nonintegrability of Θ_v or Θ_u, the main assumption of Theorem B. Therefore we have $\nabla u = \nabla v$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$. □

Proof. **(of Corollary B.1)** First we claim that both $S_{\vec{F}}(u)$ and $S_{\vec{F}}(v)$ are nowhere dense. Suppose the converse holds. Then we can find a small ball B contained in either $S_{\vec{F}}(u)$ or $S_{\vec{F}}(v)$, say $B \subset S_{\vec{F}}(u)$. That is, $\nabla u + \vec{F} = 0$ in B. It follows from $u_{IJ} = u_{JI}$ that $h_{IJ} := \partial_I F_J - \partial_J F_I = 0$ in B for all I, J. So we have $d\Theta_u = \frac{1}{2} h_{IJ} dx^I \wedge dx^J = 0 = d\Theta_v$ in B, contradicting the nonintegrability of Θ_u or Θ_v. The above argument also works for $B \subset S_{\vec{F}}(v)$. So we have shown that both $S_{\vec{F}}(u)$ and $S_{\vec{F}}(v)$ are nowhere dense in Ω.

Now we need only to show $\nabla u = \nabla v$ on $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$. We can invoke Theorem 5.1 in [9] to have $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$ (comparing

with the proof of Theorem A). By Theorem B we have $\nabla u = \nabla v$ in $\Omega \setminus [S_{\vec{F}}(u) \cup S_{\vec{F}}(v)]$.

$\square$

We remark that the condition $m \geq rank(h_{IJ}) \geq 3$ implies nonintegrability of Θ_u and Θ_v. Suppose one of them, say Θ_u, is integrable. Then (2.9) holds by Lemma 2.3. It follows from Lemma 2.1 that $rank(h_{IJ}(p)) = 0$ or 2, a contradiction. Thus we have given another proof of Theorem A by making use of Theorem B.

Proof. **(of Theorem A$'$, Corollary A$'$.1, Theorem B$'$, Corollary B$'$.1)** Note that $|\nabla u + \vec{F}| \geq C$ and $|\nabla v + \vec{F}| \geq C$ for some constant $C > 0$ imply

$$N_{\vec{F}}(u) = \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} \text{ and } N_{\vec{F}}(v) = \frac{\nabla v + \vec{F}}{|\nabla v + \vec{F}|}$$

exist. Observe that $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in Ω (a.e.) implies $\nabla N_{\vec{F}}(u) = \nabla N_{\vec{F}}(v)$ in Ω (a.e.). Moreover, noting that $\nabla|\vec{X}| = \frac{\vec{X}}{|\vec{X}|} \in L^\infty$ where $\vec{X} = (x^1, x^2, ..., x^m)$, we have

$$\partial_i |\nabla u + \vec{F}| = \sum_{j=1}^{m} \frac{(\partial_j u + F_j)\partial_i(\partial_j u + F_j)}{|\nabla u + \vec{F}|} \text{ (a.e.)}$$

by Theorem 7.8 in [15]. Now apply the same reasoning as in the proof of Theorem A (Corollary A.1, Theorem B, Corollary B.1, resp.) to reach the conclusion.

$\square$

3. Proof of Theorem C

The proof of Theorem C is similar as that of Theorem B in [9] as long as we replace "$*$" by "b" or $\vec{G}^*$ by $\vec{G}^b$. We recall the definition of $\vec{G}^b$ for $\vec{G} = (G_1, ..., G_m)$ as follows:

$$\vec{G}^b = (\sum_{k=1}^{m} a^{1k}G_k, \sum_{k=1}^{m} a^{2k}G_k, ..., \sum_{k=1}^{m} a^{mk}G_k) \tag{3.1}$$

where $a^{jk}{'}s$ are real constants such that $a^{jk} + a^{kj} = 0$ for $1 \leq j, k \leq m$. For the reader's convenience, we will sketch the idea of the proof based on some reasonings in [9].

Let $\kappa(\varepsilon)$ denote the Lebesgue measure of the set $S_{\vec{F}}(u_\varepsilon) \cap \{\nabla\varphi \neq 0\}$ where $u_\varepsilon = u + \varepsilon\varphi$, $\varphi = v - u$. There are at most countably many $\varepsilon' s$ with $\kappa(\varepsilon) > 0$ (see Section 3 in [9]). We call such an ε singular, otherwise regular (i.e., $\kappa(\varepsilon) = 0$). Now we have

Lemma 3.1 (Theorem 5.1 in [9]) *Let $u, v \in W^{1,1}(\Omega)$ be two minimizers for $\mathcal{F}_H(u)$ such that $u - v \in W_0^{1,1}(\Omega)$. Let $u_\varepsilon \equiv u + \varepsilon(v - u)$. Then for any pair of regular ε_1, $\varepsilon_2 \in [0, 1]$ (with respect to $\varphi = v - u$), there holds $N_{\vec{F}}(u_{\varepsilon_1}) = N_{\vec{F}}(u_{\varepsilon_2})$ in $\Omega \setminus [S_{\vec{F}}(u_{\varepsilon_1}) \cup S_{\vec{F}}(u_{\varepsilon_2})]$ (a.e.).*

Lemma 3.2. *Let $u, v \in W^1(\Omega)$ where the domain Ω is contained in R^m. Let $u_\varepsilon \equiv u + \varepsilon(v - u)$. Suppose $N_{\vec{F}}(u_{\varepsilon_1}) = N_{\vec{F}}(u_{\varepsilon_2})$ in $\Omega \setminus [S_{\vec{F}}(u_{\varepsilon_1}) \cup S_{\vec{F}}(u_{\varepsilon_2})]$ for a pair ε_1, ε_2 such that $\varepsilon_1 \neq \varepsilon_2$. Then for $j = 1, 2$, there holds*

$$(\nabla u_{\varepsilon_j} + \vec{F})^b \cdot (\nabla v - \nabla u) = 0 \text{ in } \Omega \text{ (a.e.)} \tag{3.2}$$

Lemma 3.2 extends Lemma 5.2 in [9]. Note that in deducing

$$N_{\vec{F}}(u_{\varepsilon_j})^b \cdot \nabla u_{\varepsilon_j} = \frac{\vec{F}^b \cdot \nabla u_{\varepsilon_j}}{|\nabla u_{\varepsilon_j} + \vec{F}|} = \vec{F}^b \cdot N_{\vec{F}}(u_{\varepsilon_j}) \tag{3.3}$$

(cf. (5.4) in the proof of Lemma 5.2 in [9]), we have used the property

$$\vec{G}^b \cdot \vec{G} = 0$$

twice. $\vec{G}^b \cdot \vec{G} = 0$ holds because

$$\begin{aligned} \vec{G}^b \cdot \vec{G} &= \sum_{j=1}^{m} a^{jk} G_k G_j \\ &= \sum_{j=1}^{m} (-a^{kj}) G_k G_j = -\vec{G}^b \cdot \vec{G}. \end{aligned}$$

Since $N_{\vec{F}}(u_{\varepsilon_1}) = N_{\vec{F}}(u_{\varepsilon_2})$ in $\Omega \backslash [S_{\vec{F}}(u_{\varepsilon_1}) \cup S_{\vec{F}}(u_{\varepsilon_2})]$ by assumption (hence $N_{\vec{F}}(u_{\varepsilon_1})^b = N_{\vec{F}}(u_{\varepsilon_2})^b$ also), we take the difference of (3.3) for $j = 1$ and $j = 2$ to obtain

$$N_{\vec{F}}(u_{\varepsilon_1})^b \cdot (\nabla u_{\varepsilon_2} - \nabla u_{\varepsilon_1}) = 0. \tag{3.4}$$

Formula (3.2) for $j = 1$ then follows from (3.4) by noting that $v - u = (u_{\varepsilon_2} - u_{\varepsilon_1})/(\varepsilon_2 - \varepsilon_1)$.

Lemma 3.3. *Let Ω be a bounded domain in R^m. Let $w \in W_0^{1,p}(\Omega)$, $\sigma \in W^{1,q}(\Omega)$, where $1 \leq p < \infty$, $q = \frac{p}{p-1}$ ($q = \infty$ for $p = 1$). Let $\vec{F}$ (a vector field) $\in W^{1,1}(\Omega) \cap L^q(\Omega)$ satisfying $div\vec{F}^b > 0$ (a.e.) or $div\vec{F}^b < 0$ (a.e.). Suppose $(\nabla\sigma + \vec{F})^b \cdot \nabla w = 0$ in Ω (a.e.). Then $w \equiv 0$ in Ω (a.e.).*

Replacing "*" by "b" in the proof of Theorem 5.3 in [9] gives a proof of Lemma 3.3. We give an outline of the proof below. Approximate w, σ, $\vec{F}$ by $\omega_j \in C_0^\infty$, $v_k \in C^\infty$, $\vec{F}_{\bar{k}} \in C^\infty$ in $W^{1,p}$, $W^{1,q}$, $W^{1,1} \cap L^q$, resp.. Suppose ω_j does not vanish identically. Then for a decreasing sequence of $a_i > 0$ converging to 0, $\Omega_{j,i} := \{|\omega_j| > a_i\} \subset\subset \Omega$ is not empty for large i and $\partial\Omega_{j,i}$ is C^∞ smooth. Consider

$$I_{j,i,k,\bar{k}} := \int_{\partial\Omega_{j,i}} |\omega_j| (\nabla v_k + \vec{F}_{\bar{k}})^b \cdot \nu$$

where ν denotes the outer normal of $\partial\Omega_{j,i}$. By using

$$\begin{aligned} \operatorname{div}(\nabla v_k + \vec{F}_{\bar{k}})^b &= \operatorname{div}(\nabla v_k)^b + \operatorname{div}(\vec{F}_{\bar{k}})^b \\ &= 0 + \operatorname{div}(\vec{F}_{\bar{k}})^b, \end{aligned}$$

we get

$$I_{j,i,k,\bar{k}} = a_i \int_{\Omega_{j,i}} \operatorname{div}(\vec{F}_{\bar{k}})^b$$

and hence

$$\lim_{i \to \infty} I_{j,i,k,\bar{k}} = 0 \tag{3.5}$$

On the other hand, we compute

$$
\begin{aligned}
(3.6) \qquad & I_{j,i,k,\bar{k}} - \int_{\Omega\setminus\{\omega_j=0\}} \{\nabla|\omega_j| \cdot (\nabla v_k + \vec{F}_{\bar{k}})^b + |\omega_j| \operatorname{div} \vec{F}^b_{\bar{k}}\} \\
&= (\int_{\Omega_{j,i}} - \int_{\Omega\setminus\{\omega_j=0\}}) \{\nabla|\omega_j| \cdot (\nabla v_k + \vec{F}_{\bar{k}})^b + |\omega_j| \operatorname{div} \vec{F}^b_{\bar{k}}\} \\
&= -\sum_{l=i}^{\infty} \int_{\Omega_{j,l+1}\setminus\Omega_{j,l}} \{\nabla|\omega_j| \cdot (\nabla v_k + \vec{F}_{\bar{k}})^b + |\omega_j| \operatorname{div} \vec{F}^b_{\bar{k}}\} \\
&= -\sum_{l=i}^{\infty} (I_{j,l+1,k,\bar{k}} - I_{j,l.k,\bar{k}}) = -\lim_{m\to\infty} I_{j,m,k,\bar{k}} + I_{j,i,k,\bar{k}}.
\end{aligned}
$$

By (3.5) and (3.6) we have

$$
\begin{aligned}
(3.7) \qquad 0 &= \int_{\Omega\setminus\{\omega_j=0\}} \{\nabla|\omega_j| \cdot (\nabla v_k + \vec{F}_{\bar{k}})^b + |\omega_j| \operatorname{div} \vec{F}^b_{\bar{k}}\} \\
&= \int_{\Omega} \{\nabla|\omega_j| \cdot (\nabla v_k + \vec{F}_{\bar{k}})^b + |\omega_j| \operatorname{div} \vec{F}^b_{\bar{k}}\}
\end{aligned}
$$

in which we have used $\nabla|\omega_j| = 0$ if $\omega_j = 0$ (p.152 in [15]). Letting $\bar{k} \to \infty$, $k \to \infty$ in (3.7) gives

$$
0 = \int_{\Omega} \{\nabla|\omega_j| \cdot (\nabla\sigma + \vec{F})^b + |\omega_j| \operatorname{div} \vec{F}^b\}.
$$

Since $(\nabla\sigma + \vec{F})^b \cdot \nabla w = 0$ by assumption, we estimate

$$
\begin{aligned}
(3.8) \qquad & \int_{\Omega} \{\nabla|\omega_j| \cdot (\nabla\sigma + \vec{F})^b \\
&= \int_{\{\omega_j>0\}} (\nabla\omega_j - \nabla w) \cdot (\nabla\sigma + \vec{F})^b - \int_{\{\omega_j<0\}} (\nabla\omega_j - \nabla w) \cdot (\nabla\sigma + \vec{F})^b \\
&\to 0 \quad \text{as } j \to \infty.
\end{aligned}
$$

On the other hand, we have

$$
(3.9) \qquad \lim_{j\to\infty} \int_{\Omega} |\omega_j| \operatorname{div} \vec{F}^b = \int_{\Omega} |w| \operatorname{div} \vec{F}^b > 0 \text{ or } < 0
$$

due to $\operatorname{div} \vec{F}^b > 0$ or < 0 (a.e.) by assumption if $w \neq 0$. In view of (3.7), (3.8), and (3.9), we reach a contradiction. Therefore $w = 0$ in Ω (a.e.).

Proof. **(of Theorem C)** The proof follows from Lemmas 3.1, 3.2, and 3.3 with $p = q = 2$, $\sigma = u_{\varepsilon_1}$, and $w = v - u$.

□

We would like to mention a result about the size of the singular set for $u \in C^1$, $\vec{F} \in C^1$ under the same condition on $\vec{F}$ as in Theorem C.

Proposition 3.4. *Let Ω be a bounded domain of R^m. Let $u \in C^1(\Omega)$, $\vec{F} = (F_1, F_2, ..., F_m) \in C^1(\Omega)$. Suppose*

$$
\operatorname{div} \vec{F}^b = \sum_{j,k=1}^{m} a^{jk} \partial_j F_k > 0 \ (< 0, \text{ resp.})
$$

where a^{jk}'s are real constants such that $a^{jk}+a^{kj}=0$. Then $S_{\vec{F}}(u)$ is nowhere dense in Ω.

Proof. Observe that $S_{\vec{F}}(u)$ is a closed set. So if $S_{\vec{F}}(u)$ is not nowhere dense in Ω, there there is a point $p_1 \in S_{\vec{F}}(u)$ such that $S_{\vec{F}}(u)$ contains $B_{r_1}(p_1)$, a ball of center p_1 with radius $r_1>0$. Take a sequence of C^∞ smooth functions u_k converging to u in C^1 norm on the closure of $B_{r_2}(p_1)$ for $0<r_2<r_1$. Let ν denote the unit outer normal. Since $\nabla u+\vec{F}=0$ in $B_{r_1}(p_1)$, we have

$$\begin{aligned} 0 &= \oint_{\partial B_{r_2}(p_1)} (\nabla u+\vec{F})^b\cdot\nu \\ &= \lim_{k\to\infty}\oint_{\partial B_{r_2}(p_1)} (\nabla u_k+\vec{F})^b\cdot\nu \\ &= \lim_{k\to\infty}\int_{B_{r_2}(p_1)} \operatorname{div}(\nabla u_k+\vec{F})^b \text{ (by the divergence theorem)} \\ &= \int_{B_{r_2}(p_1)} \operatorname{div}\vec{F}^b \text{ (since } \operatorname{div}(\nabla u_k)^b=0) \\ &> 0 \;(<0, \text{ resp.}), \end{aligned}$$

a contradiction. ∎

Note that Proposition 3.4 generalizes Lemma 3.1 in [8].

4. Proof of Theorem E and examples

Proof. **(of Theorem E)** Let $U_I=D\nu_I-F_I$. So $\nu_I=\frac{U_I+F_I}{D}$. Since ν is a unit vector, we get $D=(\sum_{I=1}^m(U_I+F_I)^2)^{1/2}$. Let $\hat{U}_I:=U_I+F_I$. By the same computation to reach (2.4), we have

$$\begin{aligned} &\delta_I\nu_J-\delta_J\nu_I \\ =\;& \frac{\partial_I\hat{U}_J-\partial_J\hat{U}_I}{D}-\frac{\hat{U}_J\hat{U}_K(\partial_I\hat{U}_K-\partial_K\hat{U}_I)}{D^3} \\ &-\frac{\hat{U}_I\hat{U}_K(\partial_K\hat{U}_J-\partial_J\hat{U}_K)}{D^3}. \end{aligned}\tag{4.1}$$

Noting that $\frac{\hat{U}_J}{D}=\nu_J$ and substituting $\hat{U}_I:=U_I+F_I$ into (4.1), we obtain

$$\begin{aligned} &\delta_I\nu_J-\delta_J\nu_I \\ =\;& \frac{U_{IJ}-\nu_J\nu_K U_{IK}-\nu_I\nu_K U_{KJ}}{D} \\ &+\frac{h_{IJ}-\nu_J\nu_K h_{IK}-\nu_I\nu_K h_{KJ}}{D} \end{aligned}\tag{4.2}$$

where $U_{IJ}:=\partial_I U_J-\partial_J U_I$ and recall $h_{IJ}=\partial_I F_J-\partial_J F_I$. By the assumption (1.6) and (4.2), we have

$$U_{IJ}-\nu_J\nu_K U_{IK}-\nu_I\nu_K U_{KJ}=0 \tag{4.3}$$

Let $U = (U_{IJ})$. Recall that we view $\nu = (\nu_J)$ as a $(m \times 1)$ unit column real vector and ν^T, the transpose of ν, as a $(1 \times m)$ unit row vector. We can write (4.3) as follows:

$$U = (U\nu)\nu^T - \nu(U\nu)^T \tag{4.4}$$

in which we have used skew-symmetry of U (i.e., $U^T = -U$ where U^T denotes the transpose of U). In terms of differential forms, we have

$$\begin{aligned} d(U_I dx^I) &= \frac{1}{2} U_{IJ} dx^I \wedge dx^J \\ &= \frac{1}{2}\{(U\nu)_I \nu_J - \nu_I (U\nu)_J\} dx^I \wedge dx^J \\ &= \frac{1}{2}\{(U\nu)^\# \wedge \nu^\# - \nu^\# \wedge (U\nu)^\#\} \\ &= (U\nu)^\# \wedge \nu^\# \end{aligned} \tag{4.5}$$

by (4.4). Recall that $w^\#$ denotes the corresponding 1-form for a vector w.

Now we observe that

$$d(D\nu^\#) = d(U_I dx^I) + dF^\#.$$

Comparing with condition (1.7), we have

$$\nu^\# \lrcorner d(U_I dx^I) = 0. \tag{4.6}$$

Substituting (4.5) into (4.6), we have

$$\begin{aligned} 0 &= \nu^\# \lrcorner ((U\nu)^\# \wedge \nu^\#) \\ &= < \nu^\#, (U\nu)^\# > \nu^\# - < \nu^\#, \nu^\# > (U\nu)^\# \\ &= -(U\nu)^\#. \end{aligned} \tag{4.7}$$

Here we have used $< \nu^\#, \nu^\# > = < \nu, \nu > = 1$ and $< \nu^\#, (U\nu)^\# > = < \nu, U\nu >$ $= 0$ since

$$\begin{aligned} < & \nu, U\nu > = < U^T \nu, \nu > \\ = & < -U\nu, \nu > = - < \nu, U\nu > . \end{aligned} \tag{4.8}$$

It follows from (4.7) that $U\nu = 0$, and hence $U = 0$ by (4.4), i.e., $U_{IJ} = 0$. This means $\partial_I U_J = \partial_J U_I$. Therefore locally we can find a (C^∞ smooth) function u such that $U_I = \partial_I u$. By the definition of U_I, we have $\nu = \frac{\nabla u + \vec{F}}{D}$. We have completed the proof of Theorem E.

□

Proposition 4.1. *Let ν be a unit vector. Condition (1.7) in Theorem E is equivalent to the following system of first order equations in D :*

$$\delta_K D = \nu_J (\partial_J \nu_K) D - \nu_J h_{JK} \tag{4.9}$$

for $1 \le K \le m$.

Proof. Observe that

$$\begin{aligned} & \nu^\# \lrcorner d(D\nu^\# - F^\#) \\ = \ & < \nu^\#, dD > \nu^\# - dD < \nu^\#, \nu^\# > + D\nu^\# \lrcorner d\nu^\# - \nu^\# \lrcorner dF^\#. \end{aligned} \tag{4.10}$$

Let D_I denote $\partial_I D$. We compute

$$\begin{aligned} & < \quad \nu^{\#}, dD > \nu^{\#} \\ = & \quad \nu_I D_I \nu_K dx^K, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \nu^{\#} \lrcorner d\nu^{\#} &= \nu_I dx^I \lrcorner ((\partial_K \nu_J) dx^K \wedge dx^J) \\ &= \nu_I (\partial_K \nu_J)(\delta^{IK} dx^J - \delta^{IJ} dx^K) \\ &= \nu_J (\partial_J \nu_K - \partial_K \nu_J) dx^K, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \nu^{\#} \lrcorner dF^{\#} &= \nu_I dx^I \lrcorner ((\partial_J F_K) dx^J \wedge dx^K) \\ &= \nu_I (\partial_J F_K)(\delta^{IJ} dx^K - \delta^{IK} dx^J) \\ &= \nu_I h_{IK} dx^K \end{aligned} \tag{4.13}$$

in which we recall that δ^{IJ} denotes the Kronecker delta and $h_{IK} := \partial_I F_K - \partial_K F_I$. Substituting (4.11), (4.12), and (4.13) into (4.10), we reduce (1.7) to the following equations:

$$\begin{aligned} 0 \quad = & \quad \nu_K \nu_I D_I - D_K \\ & + \nu_J (\partial_J \nu_K - \partial_K \nu_J) D - \nu_I h_{IK} \end{aligned} \tag{4.14}$$

for $1 \leq K \leq m$. Noting that $\delta_K := \partial_K - \nu_K \nu_I \partial_I$ and $\nu_J \partial_K \nu_J = 0$ due to $\sum_J \nu_J^2$ $= 1$ in (4.14), we get (4.9).

□

Let us discuss the (p-area) situation of dimension 2 for $\vec{F} = (-y,\, x)$. Write ν_1 $= \cos\theta$, $\nu_2 = \sin\theta$ and $\nu_\perp = \nu_2 \partial_x - \nu_1 \partial_y = \sin\theta \partial_x - \cos\theta \partial_y$. We have $d(D\nu^{\#})$ $= d(D\cos\theta dx + D\sin\theta dy) = [\partial_x(D\sin\theta) - \partial_y(D\cos\theta)]\, dx \wedge dy$, $dF^{\#} = 2dx \wedge dy$, and hence

$$\begin{aligned} & \nu^{\#} \lrcorner d(D\nu^{\#} - F^{\#}) \\ = & \quad \nu^{\#} \lrcorner \{\mathrm{div}(D\nu_\perp) - 2\} dx \wedge dy \\ = & \quad -\{\mathrm{div}(D\nu_\perp) - 2\} \nu_\perp^{\#}. \end{aligned} \tag{4.15}$$

It follows that $\nu^{\#} \lrcorner d(D\nu^{\#} - F^{\#}) = 0$ if and only if $\mathrm{div}(D\nu_\perp) - 2 = 0$ if and only if $d(D\nu^{\#} - F^{\#}) = 0$. The equation $\mathrm{div}(D\nu_\perp) - 2 = 0$ is in fact a basic (Codazzi-like) equation for a surface in 3-dimensional Heisenberg group (see, e.g., (1.17) in [8] where V is supposed to be $\nu_\perp$ here). In higher dimensions, we can have examples satisfying (1.7), but $d(D\nu^{\#} - F^{\#}) \neq 0$.

Example 4.2. In dimension $m = 4$ we take $\vec{F} = (-y^1,\, x^1,\, -y^2,\, x^2)$ (p-area situation). Let $\nu^{\#} = dx^1$ and $D = -2y^1$ (> 0 in the region of $y^1 < 0$). We compute

$$\begin{aligned} & d(D\nu^{\#} - F^{\#}) \\ = & \quad -2dy^1 \wedge dx^1 - 2(dx^1 \wedge dy^1 + dx^2 \wedge dy^2) \\ = & \quad -2dx^2 \wedge dy^2 \neq 0. \end{aligned}$$

Clearly $\nu^{\#} \lrcorner d(D\nu^{\#} - F^{\#}) = dx^1 \lrcorner (-2dx^2 \wedge dy^2) = 0$.

Example 4.3. We can have examples satisfying (1.6), but not (1.7). Take $\vec{F}$ $= 0$ and ν = a constant unit vector. So we have $\delta_I \nu_J - \delta_J \nu_I = 0$ while $h_{IJ} = \partial_I F_J - \partial_J F_I = 0$. Therefore (1.6) holds. Choose ν (constant unit) such that we can pick up another unit vector $\nu_\perp$ perpendicular to ν with the property: $(\nu_\perp)_J >$ 0 for all J. Take $D = (\nu_\perp)_J x^J$ (summation convention) > 0 in the region of all x^J > 0. It follows that $dD = (\nu_\perp)_J dx^J = \nu_\perp^\#$, and hence

$$\begin{aligned} (4.16) \qquad & < \nu^\#, dD > \\ = \ & < \nu^\#, \nu_\perp^\# > = < \nu, \nu_\perp > = 0. \end{aligned}$$

We can now compute

$$\begin{aligned} & \nu^\# \lrcorner d(D\nu^\# - F^\#) \\ = \ & \nu^\# \lrcorner d(D\nu^\#) \quad (\because \vec{F} = 0) \\ = \ & < \nu^\#, dD > \nu^\# - < \nu^\#, \nu^\# > dD \ (\because d\nu^\# = 0) \\ = \ & 0 - dD = -\nu_\perp^\# \neq 0 \ (\text{by } (4.16)). \end{aligned}$$

I.e., (1.7) does not hold. Note that for such $(\nu, D, \vec{F})$, $\nu \neq \frac{\nabla u + \vec{F}}{D}$.for any function u (recall that (1.7) is a necessary condition.for $\nu = \frac{\nabla u + \vec{F}}{D}$ for some u).

5. Appendix

In this section we collect some more facts about the properties of U satisfying (4.4) (or (4.3)). Recall that the rank of a matrix U, denoted as $rank(U)$, is the dimension of the range $Range(U)$ (or image) of U. Let $||w|| = < w, w >^{1/2}$.

Proposition A.1. *Let U be an $m \times m$ real matrix ($m \geq 2$) such that $U = -U^T$ (skew-symmetric) and $rank(U) = 2$. Then $U\nu \neq 0$ for some $\nu \neq 0$ and for such ν, we have*
(1) $U^2\nu \neq 0$;
(2) $< \nu, U\nu > = < U\nu, U^2\nu > = 0$;
(3) $Range(U)$ is spanned by $U\nu$ and $U^2\nu$;
(4) $Range(U^2)$ is also spanned by $U\nu$ and $U^2\nu$, in particular, $rank(U^2) = 2$;
(5) $U\nu$ and $U^2\nu$ are eigenvectors of U^2 with the same eigenvalue

$$\rho = -\frac{||U^2\nu||^2}{||U\nu||^2}. \tag{5.1}$$

Proof. If $U^2\nu = 0$, then

$$\begin{aligned} 0 \ &= \ < U^2\nu, \nu > \\ &= \ < U\nu, U^T\nu > = - < U\nu, U\nu > . \end{aligned}$$

So $U\nu = 0$, a contradiction. We have proved (1). Since U is skew-symmetric, we have

$$\begin{aligned} & < w, Uw > = < U^T w, w > \\ = \ & - < Uw, w > = - < w, Uw >, \text{ and hence} \end{aligned}$$

$$< w, Uw > = 0 \tag{5.2}$$

for any w. Substituting $w = \nu$ and $U\nu$, resp. in (5.2), we get (2). By (1) and (2), $U\nu$ and $U^2\nu$ form an orthogonal basis for $Range(U)$. (3) follows. Next $U^3\nu \neq 0$ by a similar argument in deducing (1). By (5.2) with $w = U^2\nu$, we get

$$< U^2\nu, U^3\nu >= 0. \tag{5.3}$$

It follows that $U^2\nu, U^3\nu$ (=$U^2(U\nu)$) are independent nonzero elements in $Range(U^2)$. On the other hand, observe that $Range(U^2) \subset Range(U)$, and hence $rank(U^2) \leq 2$. Therefore $rank(U^2) = 2$ and $Range(U^2) = Range(U)$ is also spanned by $U\nu$ and $U^2\nu$ by (3). We have proved (4). Since $U^3\nu \in Range(U^2)$ is perpendicular to $U^2\nu$ by (5.3), we conclude that

$$U^2(U\nu) = U^3\nu = \rho U\nu \tag{5.4}$$

for some $\rho \in R$. It follows that

$$\begin{aligned} \rho &< U\nu, U\nu >=< U^3\nu, U\nu > \\ &= < U^2\nu, U^T U\nu > \\ &= - < U^2\nu, U^2\nu > . \end{aligned} \tag{5.5}$$

Observe that $U^2(U^2\nu) = U(U^3\nu) = U(\rho U\nu) = \rho U^2\nu$ by (5.4). So $U\nu$ and $U^2\nu$ are eigenvectors of U^2 with the same eigenvalue ρ. Formula (5.1) follows from (5.5). We have proved (5). □

Proposition A.2. *Let U be a nonzero skew-symmetric real $m \times m$ matrix ($m \geq 2$), i.e., $U = -U^T$ and $U \neq 0$. Then $rank(U) = 2$ if and only if*

$$U = (U\nu)\nu^T - \nu(U\nu)^T \tag{5.6}$$

for any ($m \times 1$) unit column real vector ν satisfying

$$U^2\nu = \rho\nu \tag{5.7}$$

for a nonzero real number ρ.

Proof. Suppose $rank(U) = 2$. Then $Uw \neq 0$ for some $w \neq 0$. Take

$$\nu = \frac{Uw}{||Uw||}.$$

By Proposition A.1 (5) (with ν replaced by w there), we learn that ν and $U\nu$ are eigenvectors of U^2 with nonzero eigenvalue ρ (so (5.7) holds) and moreover,

$$0 \neq \rho = -\frac{||U\nu||^2}{||\nu||^2} = -||U\nu||^2. \tag{5.8}$$

By Proposition A.1 (4) and (5), we learn that 0 is the only eigenvalue different from ρ and the dimension of its eigenspace is $m-2$. Let ν_j, $j = 3, ..., m$, be orthonormal eigenvectors of U^2 with eigenvalue 0. By Proposition A.1 (1), we have $U\nu_j = 0$ (otherwise, $U^2\nu_j \neq 0$). Let

$$\tilde{U} = (U\nu)\nu^T - \nu(U\nu)^T.$$

It is now a direct verification that $\tilde{U}\nu_j = 0$, $j = 3, ..., m$, since $< \nu, \nu_j > = 0$ and $< U\nu, \nu_j > = < \nu, U^T \nu_j > = - < \nu, U\nu_j > = 0$ by $U\nu_j = 0$. On the other hand, we have

$$\begin{aligned} \tilde{U}\nu &= (U\nu) < \nu, \nu > -\nu < U\nu, \nu > \\ &= U\nu \end{aligned}$$

by $< \nu, \nu > = 1$ and $< U\nu, \nu > = 0$. We also compute

$$\begin{aligned} &\tilde{U}(U\nu) \\ = \;& U\nu < \nu, U\nu > -\nu < U\nu, U\nu > \\ = \;& 0 + U(U\nu). \end{aligned}$$

In the last equality, we have used (5.2), (5.8), and (5.7). Altogether we conclude that $\tilde{U} = U$. We have shown (5.6). The reverse direction is due to Lemma 2.1.

□

Note that Proposition A.2 includes the converse of Lemma 2.1. In the following Proposition we point out that (5.7) with ρ given by (5.8) is also a necessary condition for (5.6) to hold. Note that equation (5.6) is equivalent to

$$U - U\nu\nu^T - \nu\nu^T U = 0. \tag{5.9}$$

Let $\nu_\perp = \frac{U\nu}{||U\nu||}$. It follows from skew-symmetry of U that $< \nu_\perp, \nu > = 0$.

Proposition A.3. *Let U be a skew-symmetric real $m \times m$ matrix ($m \geq 2$) such that (5.6) (or (5.9)) holds. Then*
(1) $U^2\nu = -||U\nu||^2\nu$;
(2) $U = ||U\nu||(\nu_\perp \nu^T - \nu(\nu_\perp)^T)$.

Proof. Apply (5.6) to $U\nu$ to get $U^2\nu = U\nu < \nu, U\nu > - \nu < U\nu, U\nu > = 0 -||U\nu||^2\nu$. (1) follows. Substituting $U\nu = ||U\nu||\nu_\perp$ into (5.6) gives (2).

□

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